個體經濟學二

Microeconomics (II)

Ch8. Production Function

* **Production function-relationships between inputs (factors of production) and output**

x=f (L, K, T, …)

X：商品名稱

x： maximum amount of output that can be produced

$$\left.\begin{array}{ll} L: & \text{labor} \\ K: & \text{capital} \\ T: & \text{land} \end{array}\right\} \text{-- factors (T is fixed, 不納入討論)}$$

<u>Production period</u>（生產時程，每家廠商的 SR 和 LR 不同）

Short run (SR)：there is a fixed input → assume K is fixed in the SR

Long run (LR)：all inputs are variable

Very short run：all inputs are fixed

Very long run：f(.) can be changed (technology progress)

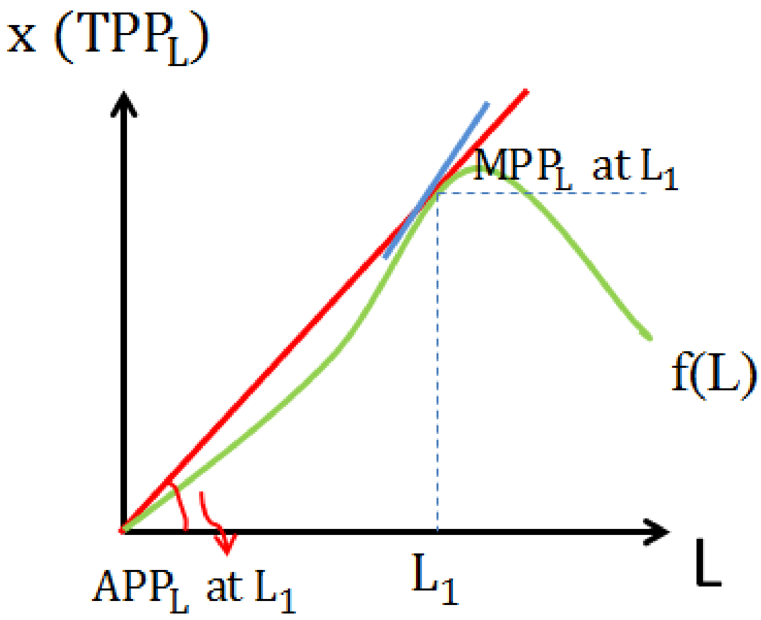
Given $K = K_0$

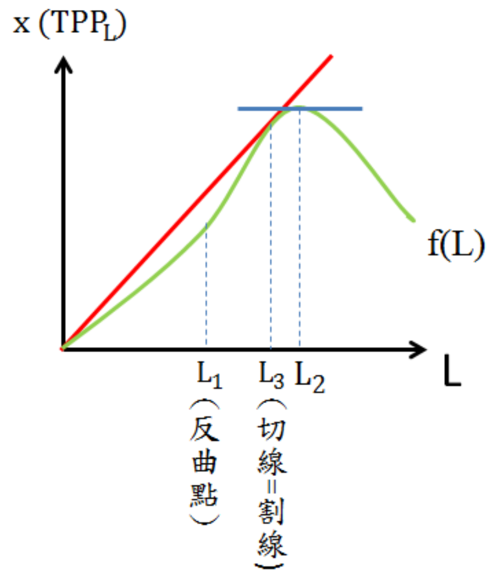
$f(L, K_0)$ is simply f(L), x=f(L)

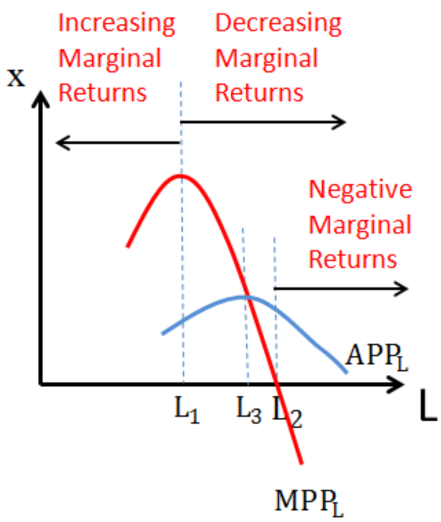
Total physical product of labor：TPP_L 總實物生產量

Marginal physical product of labor：$MPP_L = \frac{\Delta TPP_L}{\Delta L} \left(= \frac{\partial f(L,K)}{\partial L} \text{ or } f'(L) \right)$

Average physical product of labor：$APP_L = \frac{TPP_L}{L}$







利用微分證明

$$\frac{dAPP_L}{dL} = \frac{d\left(\frac{TPP_L}{L}\right)}{dL} = \frac{L\frac{dTPP_L}{dL} - TPP_L\frac{dL}{dL}}{L^2} = \frac{\frac{dTPP_L}{dL} - \frac{TPP_L}{L}}{L}$$

$$= \frac{MPP_L - APP_L}{L} > 0 \textit{ if } MPP_L > APP_L$$

$$= 0 \text{ if } MPP_L = APP_L \ \ (APP_L \text{ is max})$$

$$< 0 \textit{ if } MPP_L < APP_L$$

Similarly, given $L = L_0$

Total physical product of capital：$TPP_K = f(K, L_0) = f(K)$

Marginal physical product of capital：$MPP_K = \frac{\Delta TPP_K}{\Delta K}\left(= \frac{\partial f(L,K)}{\partial K} \text{ or } f'(K)\right)$

Average physical product of capital：$APP_K = \frac{TPP_K}{K}$

Isoquant（等產量線）

$$\mathbf{I_q(x_1) = \{(L, K) | f(L, K) = x_1\}}$$

$$x = f(L, K)$$

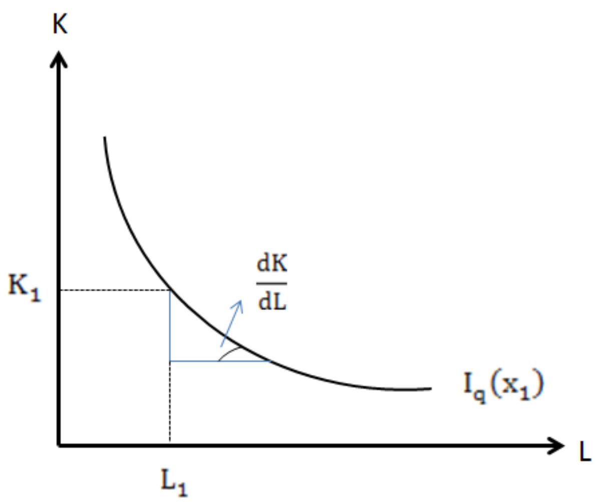
$$\Delta x = \frac{\Delta x}{\Delta L}\Delta L + \frac{\Delta x}{\Delta K}\Delta K$$

$$dx = \frac{\partial f(L,K)}{\partial L} dL + \frac{\partial f(L,K)}{\partial K} dK$$

$0 = MPP_L dL + MPP_K dK$ (Any combination of input on an isoquant produces same amount of output)

$$=> \ -\frac{dK}{dL} = \frac{MPP_L}{MPP_K}$$

$-\frac{dK}{dL}$ (slope of an isoquant): $MRTS_{LK}$ (marginal rate of technical substitution) 邊際技術替代率



Law of Diminishing Marginal Returns

$MPP_L \downarrow$ with L (if L is large enough)

$MPP_K \downarrow$ with K

We do not rule out increasing marginal returns when variable input is small

Convexity of the isoquants

Diminishing $MRTS_{LK}$

$\Rightarrow$ isoquant downward sloping ？

Diminishing marginal returns (MPP_L & MPP_K) $\overset{?}{\Rightarrow}$ Diminishing $MRTS_{LK}$

Original idea:

$$MRTS_{LK} = \frac{MPP_L}{MPP_K}$$

$L \uparrow$, $MPP_L \downarrow$ (diminishing MPP_L)
$\Rightarrow K \downarrow$ (on an isoquant)
$\Rightarrow MPP_K \uparrow$ (diminishing MPP_K)
$\Rightarrow MRTS_{LK} = \frac{MPP_L \downarrow}{MPP_K \uparrow} \downarrow$

但推論有問題，並無將 L 和 K 的交叉影響納入考慮

Simplified notation:

$$x = f(L, K)$$

$$\frac{\partial f(L, K)}{\partial L} = MPP_L = f_L$$

$$\frac{\partial f(L, K)}{\partial K} = MPP_K = f_K$$

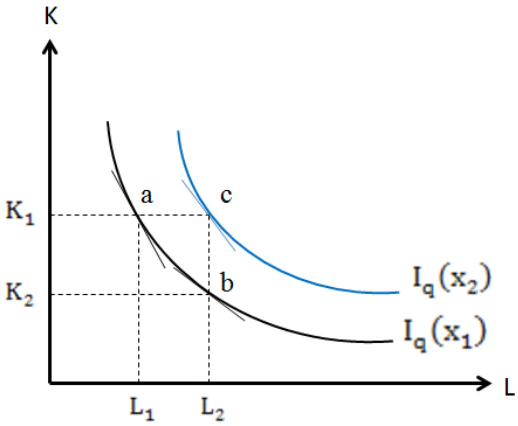
$$\frac{\partial f^2(L,K)}{\partial L^2} = \frac{\partial MPP_L}{\partial L} = f_{LL}$$

$$\frac{\partial f^2(L,K)}{\partial K^2} = \frac{\partial MPP_K}{\partial K} = f_{KK}$$

$$\frac{\partial f^2(L,K)}{\partial L\,\partial K} = \frac{\partial MPP_L}{\partial K} = f_{LK}$$

$$\frac{\partial f^2(L,K)}{\partial K\,\partial L} = \frac{\partial MPP_K}{\partial L} = f_{KL}$$

We want to compare the slope of a & the slope of b, rather than fix K and compare the slope of a & the slope of c



$$\frac{dMRTS_{LK}}{dL} = \frac{d\left(\frac{f_L}{f_K}\right)}{dL}$$

$$= \frac{f_K \frac{df_L}{dL} - f_L \frac{df_K}{dL}}{{f_K}^2}$$

$$= \frac{f_K\left(\frac{\partial f_L}{\partial L} + \frac{\partial f_L}{\partial K}\frac{dK}{dL}\right) - f_L\left(\frac{\partial f_K}{\partial L} + \frac{\partial f_K}{\partial K}\frac{dK}{dL}\right)}{{f_K}^2}$$

$$= \frac{f_K\left(f_{LL} + f_{LK}\left(-\frac{f_L}{f_K}\right)\right) - f_L\left(f_{KL} + f_{KK}\left(-\frac{f_L}{f_K}\right)\right)}{{f_K}^2} \quad \left(\text{note that}: -\frac{dK}{dL_{x=x_0}} = \frac{MPP_L}{MPP_K} = \frac{f_L}{f_K}\right)$$

$$= \frac{f_K(f_{LL} f_K - f_{LK} f_L)) - f_L(f_{KL} f_K - f_{KK} f_L)}{{f_K}^3}$$

$$= \frac{{f_K}^2 f_{LL} - f_L f_K f_{LK} - f_L f_K f_{KL} + {f_L}^2 f_{KK}}{{f_K}^3}$$

$$f_L > 0, f_K > 0, f_L^2 > 0, f_K^2 > 0, f_K^3 > 0$$

$f_{LL} < 0$, $f_{KK} < 0$ (diminishing marginal returns) if $f_{LK} > 0 \Rightarrow \dfrac{\partial MRTS_{LK}}{\partial L} < 0$

if $f_{LK} < 0 \Rightarrow \dfrac{\partial MRTS_{LK}}{\partial L} < 0$ or $\dfrac{\partial MRTS_{LK}}{\partial L} > 0$

$\therefore$ Diminishing MPP_L & MPP_K is not enough to imply diminishing $MRTS_{LK}$

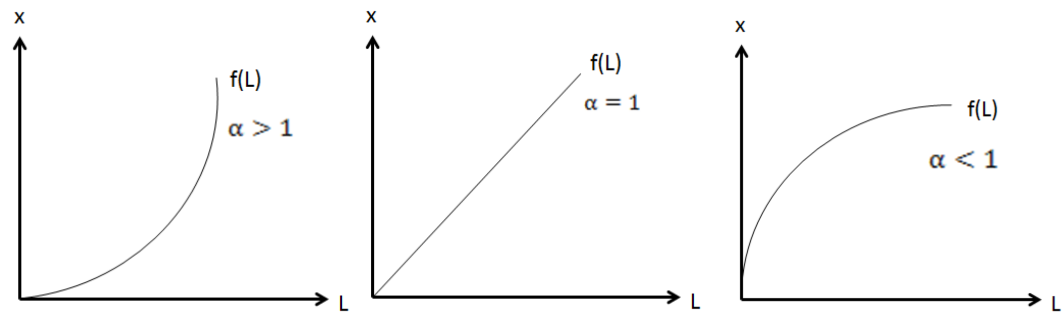
∗ Example : Cobb-Douglas production function

$x = f(L, K) = AL^{\alpha}K^{\beta}, \qquad A, \alpha, \beta > 0$

$$\left.\begin{matrix} MPP_L = \alpha AL^{\alpha-1}K^{\beta} > 0 \\ MPP_K = \beta AL^{\alpha}K^{\beta-1} > 0 \end{matrix}\right\} \therefore \text{ no ridge lines}$$

$$\frac{\partial MPP_L}{\partial L} = (\alpha - 1)\alpha AL^{\alpha-2}K^{\beta} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ if } \alpha \begin{matrix} > \\ = \\ < \end{matrix} 1 \quad \begin{matrix} \text{(increasing marginal returns)} \\ \text{(constant marginal returns)} \\ \text{(decreasing marginal returns)} \end{matrix}$$

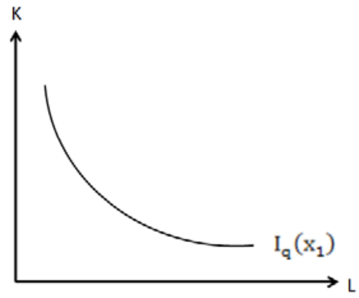
$$\frac{\partial MPP_K}{\partial K} = (\beta - 1)\beta AL^{\alpha}K^{\beta-2} \begin{matrix} > \\ = \\ < \end{matrix} 0 \text{ if } \beta \begin{matrix} > \\ = \\ < \end{matrix} 1 \quad \begin{matrix} \text{(increasing marginal returns)} \\ \text{(constant marginal returns)} \\ \text{(decreasing marginal returns)} \end{matrix}$$



<u>Indifference curve</u>

$$AL^{\alpha}K^{\beta} = x \Rightarrow K^{\beta} = \frac{x}{AL^{\alpha}}$$

$$K = \left(\frac{x}{AL^{\alpha}}\right)^{\frac{1}{\beta}} = \left(\frac{x}{A}\right)^{\frac{1}{\beta}} L^{-\frac{\alpha}{\beta}} \quad \left[\text{if } \alpha = \beta \quad K = \left(\frac{x}{A}\right)^{\frac{1}{\beta}} \frac{1}{L}\right]$$



$L \uparrow$, $K \downarrow$ isoquants are downward sloping

$$MRTS_{LK} = -\frac{dK}{dL} \left(\text{using } K = \left(\frac{x}{A}\right)^{\frac{1}{\beta}} L^{-\frac{\alpha}{\beta}} \text{ to claculate}\right)$$

$$= \frac{\alpha}{\beta}\left(\frac{x}{A}\right)^{\frac{1}{\beta}} L^{-\frac{\alpha}{\beta}-1}$$

$$MRTS_{LK} = \frac{f_L}{f_K} = \frac{\alpha A L^{\alpha-1} K^{\beta}}{\beta A L^{\alpha} K^{\beta-1}} = \frac{\alpha}{\beta}\frac{K}{L} \quad \downarrow \text{ with } L \uparrow \text{ and } K \downarrow$$

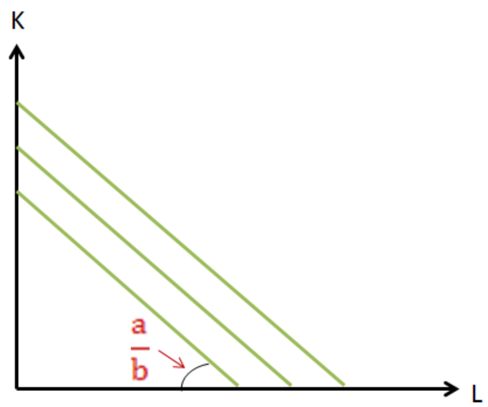
$$\frac{dMRTS_{LK}}{dL} = \frac{d\left(\frac{\alpha}{\beta}\frac{K}{L}\right)}{dL} = \frac{\alpha}{\beta}\frac{L\frac{dK}{dL} - K\frac{dL}{dL}}{L^2} = \frac{\alpha}{\beta}\frac{L\left(-\frac{\alpha}{\beta}\frac{K}{L}\right) - K}{L^2} = \frac{\alpha}{\beta}\frac{-\frac{\alpha}{\beta}K - K}{L^2} < 0$$

∴ Diminishing $MRTS_{LK}$ (在 Cobb
– Dougls 函數中一定得到 Diminishing $MRTS_{LK}$, 但不一定得到 Diminishing MPP)

Polar cases of input substitution:

1. L & K are prefect substitute inputs

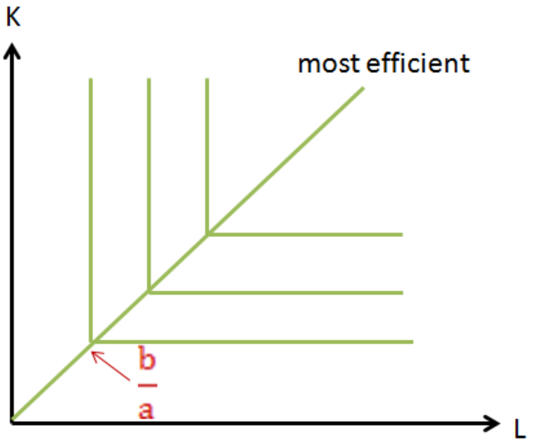
⇒ $MRTS_{LK}$ = constant ∀ all L and K



$*$ **Example:**

$$\mathbf{x = f\ (L, K) = aL\ +\ bK}$$

$$\mathrm{MRTS_{LK}} = \frac{\mathrm{MPP_L}}{\mathrm{MPP_K}} = \frac{a}{b}$$



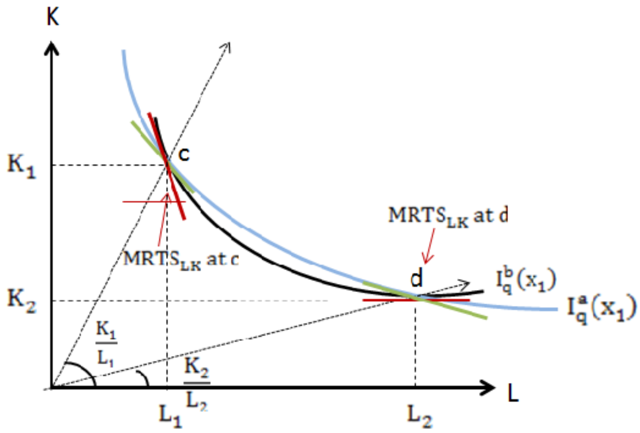
2. L & K are prefect complements in production

$*$ **Example :**

$$\mathbf{x = f\ (L, K) = min\left\{\frac{L}{a}, \frac{K}{b}\right\}}$$

most efficient L & K combination:

$$\frac{L}{a} = \frac{K}{b} \text{ or } \frac{K}{L} = \frac{b}{a}$$



衡量曲線彎曲程度⇒衡量變化後，替代能力的改變

ratio of $\frac{K}{L}$

$$\frac{K_1}{L_1} \to \frac{K_2}{L_2}$$

$I_q^a(x_1)$ $\quad$ $\mathrm{MRTS_{LK}^a}$ at c $\xrightarrow{\Delta a}$ $\mathrm{MRTS_{LK}^a}$ at d

$I_q^b(x_1)$ $\quad$ $\mathrm{MRTS_{LK}^b}$ at c $\xrightarrow{\Delta b}$ $\mathrm{MRTS_{LK}^b}$ at d

$\Delta b > \Delta a$ (substitutability between L & K, a is bettern than b)

紅線改變程度較大，b 的代替能力較差(變化越小，代表替代能力越強)

<u>Elasticity of substitution</u> (σ) 替代彈性

$\frac{K}{L} \rightarrow MRTS_{LK}$ $\quad$ $MRTS_{LK}$ depend on L and K

$$\sigma = \frac{\frac{\Delta\frac{K}{L}}{\frac{K}{L}}}{\frac{\Delta MRTS_{LK}}{MRTS_{LK}}} = \frac{d\ln\frac{K}{L}}{d\ln MRTS_{LK}}$$

分子: percentage change in K and L ratio

分母: percentage change in $MRTS_{LK}$

σ 越大, 代替能力越強, isoquant 越平緩 ⇒ 完全替代品 $\sigma \rightarrow \infty$

完全互補品 $\sigma \rightarrow 0$

＊ Example: L and K are perfect substitutes

$x = aL + bK$

$$MRTS_{LK} = \frac{MPP_L}{MPP_K} = \frac{a}{b}$$

$$\ln(MRTS_{LK}) = \ln\frac{a}{b}$$

$$\frac{1}{\sigma} = \frac{d\ln MRTS_{LK}}{d\ln\frac{K}{L}} = 0 \Rightarrow \sigma \rightarrow \infty$$

＊ Example: Cobb-Douglas production function

$x = f(L, K) = AL^{\alpha}K^{\beta}, \quad A, \alpha, \beta > 0$

$$MPP_L = \frac{\partial x}{\partial L} = A\alpha L^{\alpha-1}K^{\beta}$$

$$\frac{\partial MPP_L}{\partial L} = A\alpha(\alpha - 1)L^{\alpha-2}K^{\beta} < 0 \quad \text{if } \alpha < 1$$

Diminishing marginal returns in L

$$MPP_K = \frac{\partial x}{\partial K} = A\beta L^{\alpha}K^{\beta-1}$$

$$\frac{\partial MPP_K}{\partial K} = A\beta(\beta - 1)L^{\alpha}K^{\beta-2} < 0 \quad \text{if } \beta < 1$$

$$MRTS_{LK} = \frac{MPP_L}{MPP_K} = \frac{A\alpha L^{\alpha-1}K^{\beta}}{A\beta L^{\alpha}K^{\beta-1}} = \frac{\alpha}{\beta}\frac{K}{L}$$

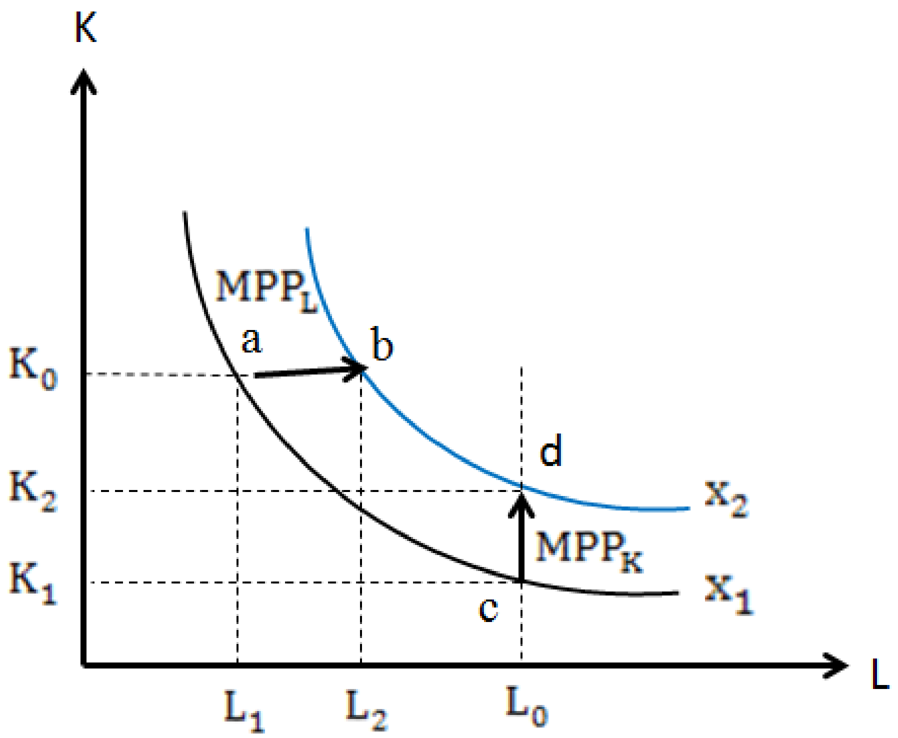
$$\ln MRTS_{LK} = \ln\left(\frac{\alpha}{\beta}\frac{K}{L}\right) = \ln\frac{\alpha}{\beta} + \ln\frac{K}{L}$$

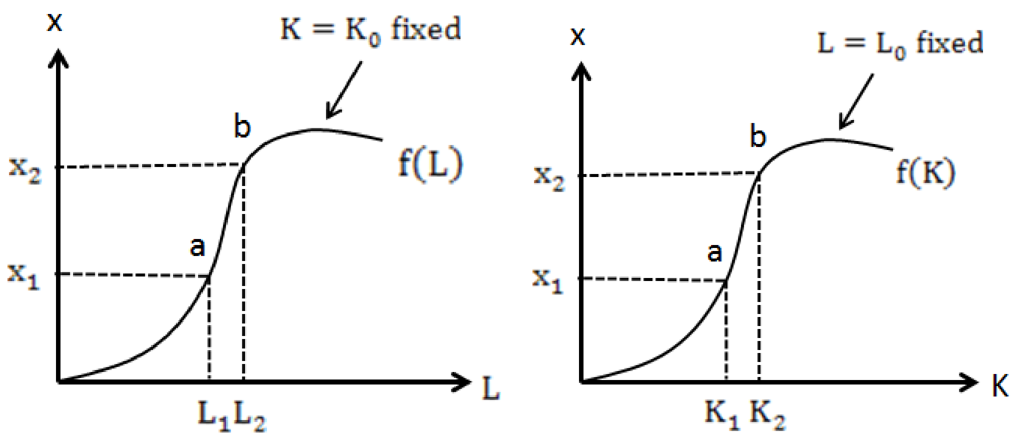
$$\frac{1}{\sigma} = \frac{d\ln MRTS_{LK}}{d\ln\frac{K}{L}} = 1 \;\Rightarrow\; \sigma = 1$$

$\therefore$ σ is constant $\Longrightarrow$ constant elasticity of substitution production function (CES)

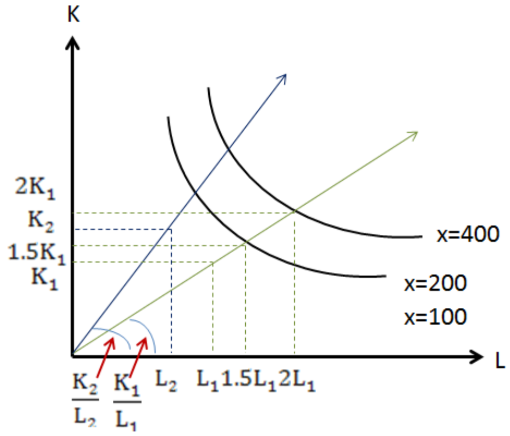
先前是 fix K，看 f(L)或 fix L，看 f(K)

若 input 同時改變時，f(L, K) ⟶ f(tL, tK) 如何改變?





Returns to scale



K 和 L 朝同一方向，同比例變動

$x = f(L, K)$

1. $f(tL, tK) = tf(L, K) = tx$

   for all L, K and $t > 0$

   constant returns to scale (c.r.t.s) 固定規模報酬

2. $f(tL, tK) > tf(L, K) = tx$ for all L, K and $t > 1$(不能縮小規模)

   increasing returns to scale (i.r.t.s.) 規模報酬遞增

3. $f(tL, tK) < tf(L, K) = tx$ for all L, K and $t > 1$

   decreasing returns to scale (d.r.t.s) 規模報酬遞減

Homogeneous function 齊次函數

$f(x_1, x_2, \cdots, x_m, \cdots, x_n)$ is homogeneous of degree **k** in $(x_1, x_2, \cdots, x_m)$ if

$$\mathbf{f(tx_1, tx_2, \cdots, tx_m, x_{m+1}, x_{m+2} \cdots, x_n) = t^k f(x_1, x_2, \cdots, x_m, \cdots, x_n)}, t > 0$$

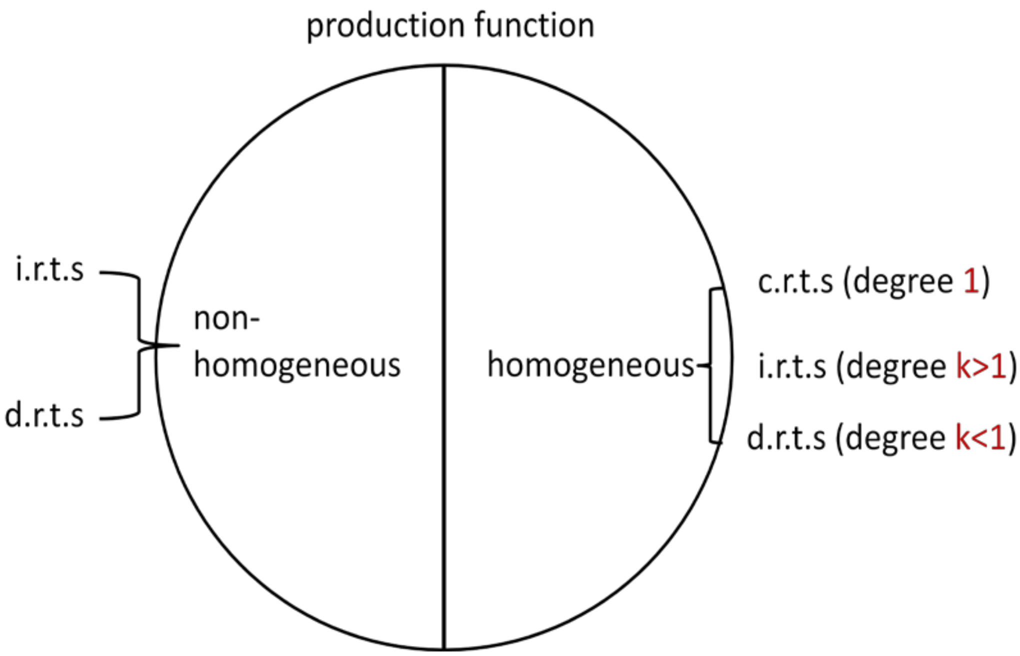
$x = f(L, K)$ is homogeneous of degree ***k*** in L & K if $f(tL, tK) = t^k f(L, K)$

constant returns to scale $\Longrightarrow$ $f(L, K)$ is homogeneous of degree **1** in L & K

However, production function may not be homogeneous.

Non − homogeneous $\leftrightarrow$ returns to scale

兩者是不同的觀念，除非是 c.r.t.s 一定是 homogeneous of degree 1



∗ **Example:**

$f(L, K) = L + K^2$ is not a homogeneous function

$f(tL, tK) = tL + (tK)^2 = t(L + tK^2) > tL + tK^2$ if $t > 1$

$\Rightarrow$ increasing returns to scale

Given a production function is homogeneous of degree k in L and K,

i.e. $f(tL, tK) = t^k f(L, K)$

$k > 1 \Rightarrow f(tL, tK) = t^k f(L, K) > tf(L, K)$ for $t > 1$

$\Rightarrow f(L, K)$ is increasing returns to scale

$k < 1 \Rightarrow f(tL, tK) = t^k f(L, K) < tf(L, K)$ for $t > 1$

$\Rightarrow f(L, K)$ is decreasing returns to scale

Given a homogeneous production function $f(tL, tK) = t^k f(L, K)$,

if $f(L, K)$ is increasing returns to scale, i.e. $f(tL, tK) > tf(L, K)$ for all $t > 1$

$\Rightarrow$ imply $k > 1$

if $f(L, K)$ is decreasing returns to scale, i.e. $f(tL, tK) < tf(L, K)$ for all $t > 1$

$\Rightarrow$ imply $k < 1$

∗ Example:

$$\mathbf{x = f(L, K) = AL^{\alpha}K^{\beta}}$$

$$f(tL, tK) = A(tL)^{\alpha}(tK)^{\beta}$$

$$= t^{\alpha+\beta} AL^{\alpha}K^{\beta}$$

$$= t^{\alpha+\beta} f(L, K)$$

$$\alpha + \beta = 1 \Longrightarrow f(tL, tK) = tf(L, K)$$

$$\Longrightarrow f(L, K) \text{ is constant returns to scale}$$

$$\alpha + \beta > 1 \Longrightarrow f(tL, tK) > tf(L, K) \ \ for\ t > 1$$

$$\Longrightarrow f(L, K) \text{ is increasing returns to scale}$$

$$\alpha + \beta < 1 \Longrightarrow f(tL, tK) < tf(L, K) \ \ for\ t > 1$$

$$\Longrightarrow f(L, K) \text{ is decreasing returns to scale}$$

∗ Two polar cases

Case1: L and K are perfect substitutes

$x = f(L, K) = aL + bK \qquad \Longrightarrow$ constant

$x = f(L, K) = (aL + bK)^3 \qquad \Longrightarrow$ incresing

$x = f(L, K) = (aL + bK)^{0.5} \qquad \Longrightarrow$ decresing

Case 2: L and K are perfect complements

$x = f(L, K) = \min\{\frac{L}{a}, \frac{K}{b}\} \quad \Longrightarrow$ constant

$x = f(L, K) = (\min\{\frac{L}{a}, \frac{K}{b}\})^3 \quad \Longrightarrow$ incresing

$x = f(L, K) = (\min\{\frac{L}{a}, \frac{K}{b}\})^{0.5} \quad \Longrightarrow$ decresing

$\ast$ Example: Cobb-Douglas production function

$x = f(L, K) = AL^{\alpha}K^{\beta}$

$f(tL, tK) = A(tL)^{\alpha}(tK)^{\beta}$

$= t^{\alpha+\beta} AL^{\alpha}K^{\beta}$

$= t^{\alpha+\beta} f(L, K) \quad \Longrightarrow AL^{\alpha}K^{\beta}$ is homogeneous of degree $\alpha + \beta$ in L and K

$\ast$ Example: $f(L, K) = (aL + bK)^3$

$f(tL, tK) = (a(tL) + b(tK))^3$

$= t^3(aL + bK)^3$

$= t^3 f(L, K) \quad \Longrightarrow (atL + btK)^3$ is homogeneous of degree 3 in L and K

$\ast$ Example : $x = f(L, K) = L^{\alpha} + K^{\beta}$

$f(tL, tK) = (tL)^{\alpha} + (tK)^{\beta} = t^{\alpha}(L^{\alpha} + t^{\beta-\alpha}K^{\beta})$

① if $\alpha = \beta$, $f(tL, tK) = t^{\alpha}(L^{\alpha} + K^{\beta})$

$f(L, K)$ is homogeneous of degree α in L and K

② if $\alpha \neq \beta, f(tL, tK) = t^{\alpha}(L^{\alpha} + t^{\beta-\alpha}K^{\beta}) \neq t^{\alpha}(L^{\alpha} + K^{\beta})$

$f(L, K)$ is not a homogeneous function

①$\alpha \neq \beta$, suppose $\alpha > \beta$

$f(tL, tK) = t^{\alpha}(L^{\alpha} + \frac{1}{t^{\alpha-\beta}}K^{\beta})$ $< K^{\beta}$

Given $t > 1, t^{\alpha-\beta} > 1$

$\Longrightarrow f(tL, tK) < t^{\alpha}\left(L^{\alpha} + K^{\beta}\right) = t^{\alpha}f(L, K)$

$\alpha < 1 \quad f(tL, tK) < tf(L, K)$ decreasing returns to scale

②$\alpha \neq \beta$, suppose $\alpha < \beta$

$f(tL, tK) = t^{\alpha}(L^{\alpha} + t^{\beta-\alpha}K^{\beta})$ $> K^{\beta}$

Given $t > 1, t^{\beta-\alpha} > 1$

$\Longrightarrow f(tL, tK) > t^{\alpha}\left(L^{\alpha} + K^{\beta}\right)$

$\alpha > 1 \quad f(tL, tK) > tf(L, K)$ increasing returns to scale

Homogeneous function vs MRTS

A production function $f(L, K)$ is homogeneous of degree k in L and K,

i.e. $f(tL, tK) = t^{k}f(L, K)$, are $f_L(= MPP_L)$ and $f_K(= MPP_K)$ also homogeneous?

$\Longrightarrow f_L(L, K)$ and $f_K(L, K)$ are homogeneous of degree $k - 1$ in L and K

(A homogeneous function of degree k has homogeneous of degree k − 1 first derivatives)

(proof)

$f(tL, tK) = t^{k}f(L, K) \quad t \neq 0$ for all L, K

$$f_L(tL, tK) = \frac{\partial f(tL, tK)}{\partial L} = \frac{\partial t^{k}f(L, K)}{\partial L}$$

$$\frac{\partial f(tL,tK)}{\partial L} = \frac{\partial f(tL,tK)}{\partial tL} \bullet \frac{dtL}{dL}(\text{by chain} - \text{rule}) = f_L(tL,tK) \bullet t$$

$$\frac{\partial t^k f(L,K)}{\partial L} = t^k f_L(L,K)$$

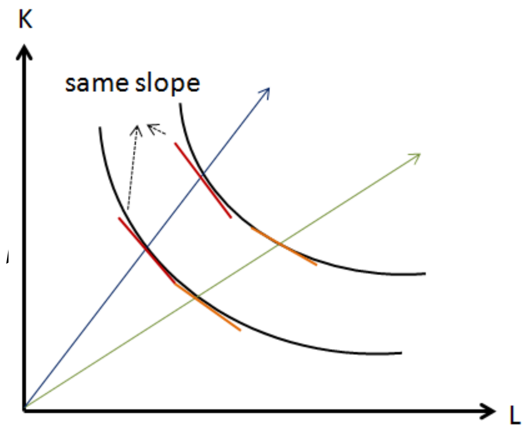
$$f_L(tL,tK) \bullet t = t^k f_L(L,K)$$

$$\Longrightarrow f_L(tL,tK) = t^{k-1} f_L(L,K)$$

$\Longrightarrow$ $f_L(L,K)$ is homogeneous of degree $k-1$ in L and K

Similar proof for $f_K(tL,tK)$

$f_K(tL,tK) = t^{k-1} f_K(L,K) \Longrightarrow$
$f_K(L,K)$ is homogeneous of degree $k-1$ in L and K



$$MRTS_{LK}(tL,tK) = \frac{MPP_L(tL,tK)}{MPP_K(tL,tK)} = \frac{t^{k-1} MPP_L(L,K)}{t^{k-1} MPP_K(L,K)}$$

$$= \frac{MPP_L(L,K)}{MPP_K(L,K)} = MRTS_{LK}(L,K)$$

$$= t^0 MRTS_{LK}(L,K)$$

$\Longrightarrow$ $MRTS_{LK}(L,K)$ is homogeneous of degree 0 in L and K

$\therefore$ $MRTS_{LK}(L,K)$ depends on $\frac{K}{L}$ only（只要知道一條等產量線，就可以推導其他條等產量線，因為斜率一樣)

<u>Homothetic</u> （**位似**）

$f(L, K)$ is homothetic if it is a positive transformation of a homogeneous function

$\mathbf{f(L,K) = g\big(h(L,K)\big)}$, $h(L, K)$ is homogeneous of degree k in L & K, $g'(.) > 0$, f *is not necessarily homogeneous*

$f(L, K)$ is homothetic and has the form of $f(L, K) = g\big(h(L, K)\big)$

$$\mathrm{MRTS_{LK}}(L, K) = \frac{f_L(L, K)}{f_K(L, K)} = \frac{g' h_L(L, K)}{g' h_K(L, K)} = \frac{h_L(L, K)}{h_K(L, K)}$$

$$\mathrm{MRTS_{LK}}(tL, tK) = \frac{f_L(tL, tK)}{f_K(tL, tK)} = \frac{\frac{\partial f}{\partial L}(tL, tK)}{\frac{\partial f}{\partial K}(tL, tK)} = \frac{g' h_L(tL, tK)}{g' h_K(tL, tK)} = \frac{t^{k-1} h_L(L, K)}{t^{k-1} h_K(L, K)}$$

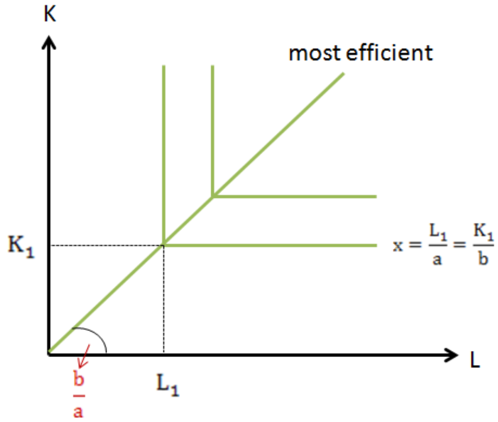
$= \mathrm{MRTS_{LK}}(L, K)$ ∴ homothetic 也有類似的特性

∗ Two polar cases

Case1: L and K are perfect complements in production

$$x = f(L, K) = \min\left\{\frac{L}{a}, \frac{K}{b}\right\}$$

most efficient L & K combination: $\frac{L}{a} = \frac{K}{b}$ or $\frac{K}{L} = \frac{b}{a}$

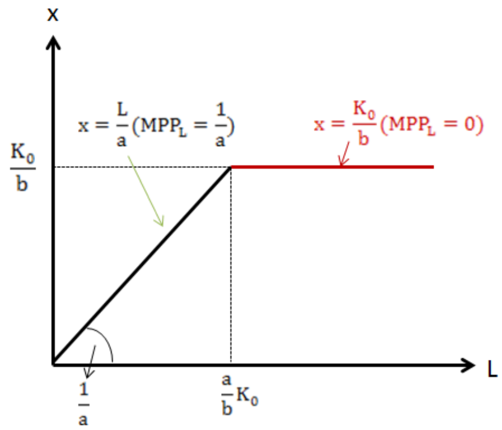


Given $K = K_0$

K fixed ∴ 當 $\frac{L}{a}$ 超過 $\frac{K_0}{b}$ 時, 便無法再生產

$$\frac{L}{a} > \frac{K_0}{b} \Longrightarrow MPP_L = 0$$

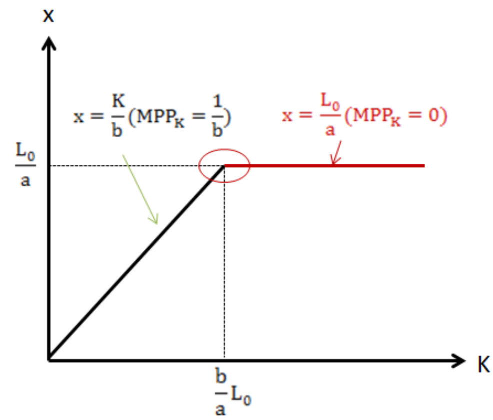
$$\Longrightarrow L > \frac{a}{b}K_0, \qquad x = \frac{L}{a} = \frac{\frac{a}{b}K_0}{a} = \frac{K_0}{b}$$



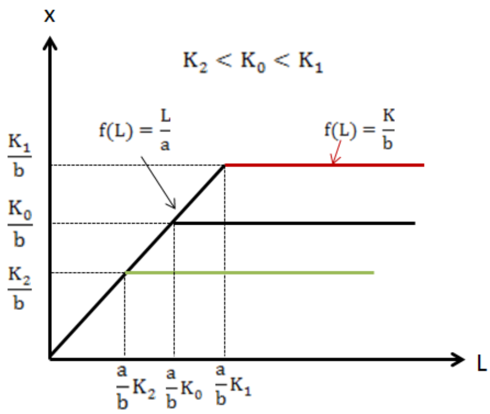
Given $L = L_0$

$$\frac{K}{b} > \frac{L_0}{a} \Longrightarrow MPP_K = 0$$

$$\Longrightarrow K > \frac{b}{a}L_0, \qquad x = \frac{K}{b} = \frac{\frac{b}{a}L_0}{b} = \frac{L_0}{a}$$



Elasticity of substitution $= \frac{1}{\infty} \rightarrow 0$



$$K = K_0, \qquad x = \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = \frac{L}{a} \text{ if } \frac{L}{a} \le \frac{K_0}{b} \text{ or } L \le \frac{a}{b} K_0$$

$$\Longrightarrow MPP_L = \frac{\partial f(L,K)}{\partial L} = \frac{1}{a}$$

$$x = \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = \frac{K_0}{b} \text{ if } \frac{L}{a} > \frac{K_0}{b} \text{ or } L > \frac{a}{b} K_0$$

$$\Longrightarrow MPP_L = \frac{\partial f(L,K)}{\partial L} = 0$$

Similarly, given $L = L_0$

$$x = \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = \frac{K}{b} \text{ if } \frac{K}{b} \le \frac{L_0}{a} \text{ or } K \le \frac{b}{a} L_0$$

$$\Longrightarrow MPP_K = \frac{\partial f(L,K)}{\partial K} = \frac{1}{b}$$

$$x = \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = \frac{L_0}{a} \text{ if } \frac{K}{b} > \frac{L_0}{a} \text{ or } K > \frac{b}{a} L_0$$

$$\Longrightarrow MPP_L = \frac{\partial f(L,K)}{\partial K} = 0$$

$$f(tL, tK) = \min\left\{\frac{tL}{a}, \frac{tK}{b}\right\} = t \min\left\{\frac{L}{a}, \frac{K}{b}\right\} = tf(L,K)$$

$\therefore$ homogeneous of degree 1 in L & K $\Longrightarrow$ constant returns to scale

∗ Example : $x = f(L,K) = (\min\{\frac{L}{a}, \frac{K}{b}\})^r$

$$f(tL, tK) = (\min\{\frac{tL}{a}, \frac{tK}{b}\})^r = t^r(\min\{\frac{L}{a}, \frac{K}{b}\})^r = t^r f(L,K)$$

∴ homogeneous of degree r in L and K

⟹ L & K are perfect complements

$0 < r < 1$, decreasing returns to scale

$r = 1$, constant returns to scale

$r > 1$, increasing returns to scale

$$f(L,K) = A + \min\left\{\frac{L}{a}, \frac{K}{b}\right\},$$ A, a, b are some positive numbers

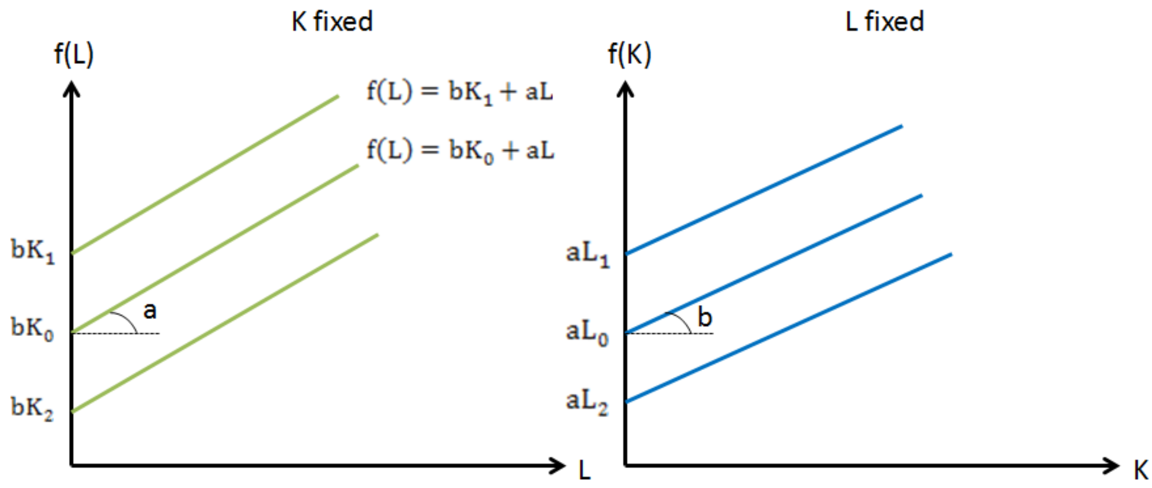
$$f(tL, tK) = A + \min\left\{\frac{tL}{a}, \frac{tK}{b}\right\} = A + t\min\left\{\frac{L}{a}, \frac{K}{b}\right\} \neq tf(L,K)$$

∴ it is not a homogeneous function, L & K are perfect complements

Case2: L and K are perfect substitutes

$f(L,K) = aL + bK$

$$\left.\begin{array}{l} MPP_L = a \\ MPP_K = b \end{array}\right\}$$ fixed coefficient production function



Isoquant

$\{(L, K)|f(L, K) = aL + bK = x_1\} = I_q(x_1)\ \ x_2 < x_1 < x_3$

$MRTS_{LK}(L, K) = \frac{MPP_L(L, K)}{MPP_K(L, K)} = \frac{a}{b}$ consatnt(no L and K)

$f(tL, tK) = a(tL) + b(tK) = t(aL + bK) = tf(L, K)$

$\therefore$ is homogeneous of degree 1 in L and K $\Longrightarrow$ CRS

$MRTS_{LK}(tL, tK) = MRTS_{LK}(L, K)$

$\therefore$ $MRTS_{LK}(L, K)$ is homogeneous of degree 0 in L and K

$f(L, K) = (aL + bK)^r$ is homogeneous of degree r in L and K

$$\begin{cases} r = 1 \Longrightarrow \mathrm{CRS} \\ r > 1 \Longrightarrow IRS \\ r < 1 \Longrightarrow DRS \end{cases}$$

However, f(L, K)=A+(aL+bK) is not homogeneous, L & K are still perfect substitutes

$\sigma \to \infty$ $\because$ perfect substitute

$*$ Example : $\mathbf{f(L, K) = (aL + bK)^2}$

$$MRTS_{LK}(L, K) = \frac{2(aL + bK) \cdot a}{2(aL + bK) \cdot b} = \frac{a}{b}$$

$$MRTS_{LK}(tL, tK) = \frac{2(atL + btK) \cdot at}{2(atL + btK) \cdot bt} = \frac{a}{b} = MRTS_{LK}(L, K)$$

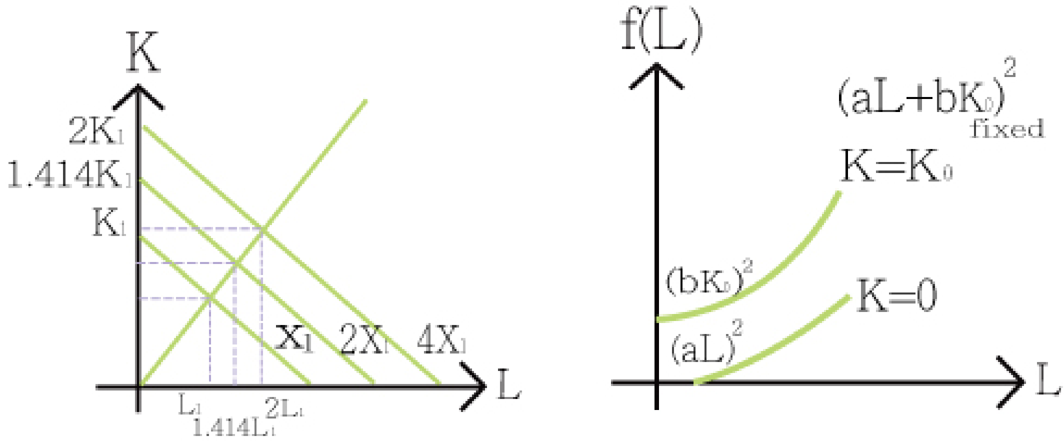


Figure 50:

$$f(L, K) = (aL + bK)^2$$

$$f(tL, tK) = (atL + btK)^2 = t^2(aL + bK)^2 = t^2 f(L, K)$$

$\Longrightarrow$homogeneous of degree 2 in L and K

$\Longrightarrow$increasing returns to scale

$$f_L = MPP_L = 2(aL + bK) \cdot a$$

$f_{LL} = \dfrac{\partial MPP_L}{\partial L} = 2a^2 > 0$ increasing marginal return in L (and also in K)

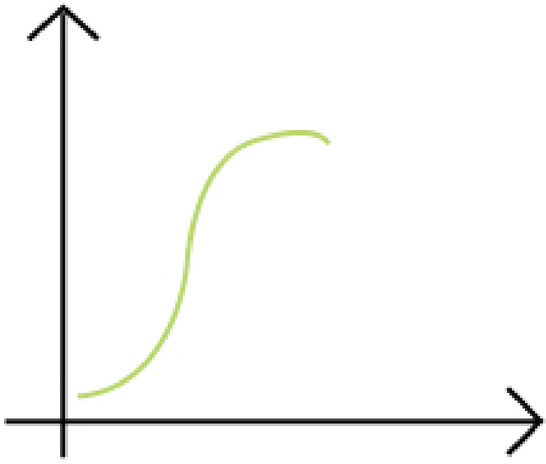


Figure 51:

$*$ **Example : $\mathbf{f(L, K) = (aL + bK)^{0.5}}$**

check：

decreasing returns to scale

diminishing marginal return

$*$ **Constant Elasticity of Substitution production function , CES**

$*$ **Cobb-Douglas production function**

$x = f(L, K) = AL^{\alpha}K^{\beta}$

$$MRTS_{LK}(L, K) = \frac{MPP_L(L, K)}{MPP_K(L, K)} = \frac{\alpha AL^{\alpha-1}K^{\beta}}{\beta AL^{\alpha}K^{\beta-1}} = \frac{\alpha}{\beta}\frac{K}{L}$$

$$\sigma = \frac{d\ln\frac{K}{L}}{d\ln MRTS_{LK}}$$

$$\ln MRTS_{LK} = \ln\left(\frac{\alpha}{\beta}\frac{K}{L}\right) = \ln\frac{\alpha}{\beta} + \ln\frac{K}{L}$$

$$\frac{1}{\sigma} = \frac{d\ln MRTS_{LK}}{d\ln\frac{K}{L}} = 1 \;\Rightarrow\; \sigma = 1 \text{ consatnt}$$

CES production function:

$x = f(L, K) = (L^{\rho} + K^{\rho})^{\frac{\varepsilon}{\rho}} \quad , \varepsilon > 0, 0 < \rho < 1$

$f(tL, tK) = ((tL)^{\rho} + (tK)^{\rho})^{\frac{\varepsilon}{\rho}} = (t^{\rho}(L^{\rho} + K^{\rho}))^{\frac{\varepsilon}{\rho}} = t^{\varepsilon}(L^{\rho} + K^{\rho})^{\frac{\varepsilon}{\rho}} = t^{\varepsilon}f(L, K)$

$\therefore$ $f(L, K)$ is homogeneous of degree $\boldsymbol{\varepsilon}$ in L and K

$\varepsilon < 1$, decreasing returns to scale

$\varepsilon = 1$, constant returns to scale

$\varepsilon > 1$, increasing returns to scale

$\sigma = ?$

$$MRTS_{LK}(L, K) = \frac{MPP_L(L, K)}{MPP_K(L, K)} = \frac{\frac{\varepsilon}{\rho}(L^{\rho} + K^{\rho})^{\frac{\varepsilon}{\rho}-1}\rho L^{\rho-1}}{\frac{\varepsilon}{\rho}(L^{\rho} + K^{\rho})^{\frac{\varepsilon}{\rho}-1}\rho K^{\rho-1}} = (\frac{L}{K})^{\rho-1} = (\frac{K}{L})^{1-\rho}$$

$$\ln MRTS_{LK} = \ln(\frac{K}{L})^{1-\rho} = (1-\rho)\ln\frac{K}{L}$$

$$\frac{d \ln MRTS_{LK}}{d \ln \frac{K}{L}} = 1 - \rho$$

$$\therefore \sigma = \frac{1}{1-\rho} = \text{constant}$$

$\rho \to 0$ 時, CES production function 的極限是 Cobb − Douglas production function

等產量線為直線時 $\sigma = \infty \;\; \therefore \rho = 1$